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An Application of Ridge Regression in Variance Component Estimation

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Abstract

One approach to estimating variance components associated with mixed effects models is to first estimate the fixed effects and then use these estimates in the subsequent estimation of the components. The presence of multicollinearity may cause the variance of the estimates of fixed effects parameters to be large, which in turn may lead to poor estimates of the components. Ridge regression methods attempt to improve the estimates of the fixed effects parameters and have been widely studied. This paper reports the results of an investigation with emphasis on the goodness of the variance components estimates following use of ridge regression methods to estimate the fixed effects.

KEY WORDS

Variance components

Ridge regression

Multicollinearity

Mixed model

Mean square error

1. THE PROBLEM AND A SUGGESTED METHOD OF SOLUTION

We consider the mixed effects linear model

$$\underset{\sim}{Y} = \underset{\sim}{X}\underset{\sim}{\tau} + \underset{\sim}{Z}\underset{\sim}{B} + \underset{\sim}{E} \tag{1}$$

where

$\underset{\sim}{Y}$ is an $n \times 1$ vector of observations;

$\underset{\sim}{X}$ is an $n \times t$ matrix of auxiliary data where a high level of multicollinearity exists among the columns of $\underset{\sim}{X}$;

$\underset{\sim}{\tau}$ is a $t \times 1$ vector of fixed effects parameters;

$\underset{\sim}{Z}$ is an $n \times b$ design matrix of 0's and 1's;

$\underset{\sim}{B}$ is a $b \times 1$ vector of random variables, $\underset{\sim}{B} \sim \underset{\sim}{N}_b(\underset{\sim}{\phi}, \sigma_b^2 \underset{\sim}{I})$, $\underset{\sim}{\phi}$ being the null vector;

and

$\underset{\sim}{E}$ is an $n \times 1$ vector of random variables, $\underset{\sim}{E} \sim \underset{\sim}{N}_n(\underset{\sim}{\phi}, \sigma_e^2 \underset{\sim}{I})$.

Our objective is to obtain precise estimates of the variance components, σ_b^2 and σ_e^2, where precision is measured by mean square error. A method for accomplishing this objective is to estimate the vector $\underset{\sim}{\tau}$, adjust the observation vector $\underset{\sim}{Y}$ for the fixed effects and then estimate the variance components from the adjusted data. An approach to estimating $\underset{\sim}{\tau}$ is to consider the random effects as fixed effects during the estimation of $\underset{\sim}{\tau}$. We adjoin the matrices $\underset{\sim}{X}$ and $\underset{\sim}{Z}$ to form $\underset{\sim}{\mathcal{X}} = (\underset{\sim}{X}, \underset{\sim}{Z})$ and we adjoin the vectors $\underset{\sim}{\tau}'$ and $\underset{\sim}{B}'$ to create $\underset{\sim}{\xi}' = (\underset{\sim}{\tau}', \underset{\sim}{B}')$. A fixed effects model can then be written as

$$\underset{\sim}{Y} = \underset{\sim}{\mathcal{X}}\underset{\sim}{\xi} + \underset{\sim}{E} \quad .$$

$\underset{\sim}{\xi}$ can be estimated in the standard least squares manner and then

$$\hat{\underset{\sim}{\tau}} = \underset{\sim}{A}\hat{\underset{\sim}{\xi}} = [\underset{\sim}{I}, \underset{\sim}{\phi}](\underset{\sim}{\mathcal{X}}'\underset{\sim}{\mathcal{X}})^{-1}\underset{\sim}{\mathcal{X}}'\underset{\sim}{Y}; \text{ where } \underset{\sim}{A} = [\underset{\sim}{I}, \underset{\sim}{\phi}] \ , \tag{2}$$

where the $\underset{\sim}{I}$ component of $\underset{\sim}{A}$ is $t \times t$, and $\underset{\sim}{\phi}$ is $t \times b$.

Under the assumption of a fixed effects model

$$\underset{\sim}{MSE}(\hat{\underset{\sim}{\tau}}) = \mathcal{E}(\hat{\underset{\sim}{\tau}} - \underset{\sim}{\tau})'(\hat{\underset{\sim}{\tau}} - \underset{\sim}{\tau}) \geq \sigma_e^2/\gamma_{min} \tag{3}$$

where γ_{min} is the minimum characteristic root of $\underset{\sim}{\mathcal{X}}'\underset{\sim}{\mathcal{X}}$. This root can be near zero because of the presence of multicollinearity and consequently the $\underset{\sim}{MSE}(\hat{\underset{\sim}{\tau}})$ can be large. Poor estimates of $\underset{\sim}{\tau}$ may in turn produce poor estimates of the variance components. We thus propose to modify the estimator of $\underset{\sim}{\tau}$ and we choose to do it in the spirit of Hoerl and Kennard [1,2].

A ridge regression type estimator of $\underset{\sim}{\tau}$ can take the form $\hat{\underset{\sim}{\tau}}_k = \underset{\sim}{L}_k\underset{\sim}{Y}$ where $\underset{\sim}{L}_k = \underset{\sim}{A}\underset{\sim}{M}_k\underset{\sim}{\mathcal{X}}'$, $\underset{\sim}{M}_k = (\underset{\sim}{\mathcal{X}}'\underset{\sim}{\mathcal{X}} + k\underset{\sim}{I})^{-1}$ and where k is a suitably chosen small positive constant. It is well known that for a fixed effects model $\underset{\sim}{MSE}(\hat{\underset{\sim}{\tau}}_k) < \underset{\sim}{MSE}(\hat{\underset{\sim}{\tau}}_0)$ for certain values of k . This inequality is a consequence of the fact that the minimum characteristic root of $(\underset{\sim}{\mathcal{X}}'\underset{\sim}{\mathcal{X}} + k\underset{\sim}{I}) > \gamma_{min}$.

We now turn our attention to the estimation of σ_b^2 and σ_e^2 in the model of display (1). By subtracting $\underset{\sim}{X}\hat{\underset{\sim}{\tau}}_k$ from each side of equation (1) we obtain

$$\underset{\sim}{W}_k = (\underset{\sim}{I} - \underset{\sim}{X}\underset{\sim}{L}_k)\underset{\sim}{Y} = (\underset{\sim}{I} - \underset{\sim}{X}\underset{\sim}{L}_k)\underset{\sim}{X}\underset{\sim}{\tau} + (\underset{\sim}{I} - \underset{\sim}{X}\underset{\sim}{L}_k)\underset{\sim}{Z}\underset{\sim}{B} + (\underset{\sim}{I} - \underset{\sim}{X}\underset{\sim}{L}_k)\underset{\sim}{E} \ .$$

Letting $\underset{\sim}{G}_k = \underset{\sim}{I} - \underset{\sim}{X}\underset{\sim}{L}_k$ we can write

$$\underset{\sim}{W}_k = \underset{\sim}{G}_k\underset{\sim}{X}\underset{\sim}{\tau} + \underset{\sim}{G}_k\underset{\sim}{Z}\underset{\sim}{B} + \underset{\sim}{G}_k\underset{\sim}{E} \ . \tag{4}$$

For $k = 0$, the coefficient matrix of $\underset{\sim}{\tau}$ in this model is identically zero and Monte Carlo investigations indicate that this is approximately the case for small positive values of k . We therefore drop the $\underset{\sim}{\tau}$ term from the model. Similarly, we assert that $\underset{\sim}{G}_k\underset{\sim}{Z}$ is approximately $\underset{\sim}{Z}$ for small values of k and we replace $\underset{\sim}{G}_k\underset{\sim}{Z}$ by $\underset{\sim}{Z}$. We denote the vector of errors by

$\underset{\sim}{F}_k = \underset{\sim}{G}_k\underset{\sim}{E}$. Note that the random variables in $\underset{\sim}{F}_k$ are dependent. The modified model of display (4) thus takes the form

$$\underset{\sim}{W}_k = \underset{\sim}{Z}\underset{\sim}{B} + \underset{\sim}{F}_k$$

where only $\underset{\sim}{F}_k$ and $\underset{\sim}{W}_k$ depend upon k.

In order to estimate σ_b^2 and σ_e^2 we consider the classical quadratic forms $\underset{\sim}{W}_k'[\underset{\sim}{Z}(\underset{\sim}{Z}'\underset{\sim}{Z})^{-1}\underset{\sim}{Z}' - \underset{\sim}{J}/n]\underset{\sim}{W}_k$ and $\underset{\sim}{W}_k'[\underset{\sim}{I} - \underset{\sim}{Z}(\underset{\sim}{Z}'\underset{\sim}{Z})^{-1}\underset{\sim}{Z}']\underset{\sim}{W}_k$. Here $\underset{\sim}{J}$ denotes the usual n × n matrix of 1's. Because the error vector $\underset{\sim}{F}_k$ has dependent variables, the expectations of the above quadratic forms are not the usual expressions in σ_b^2 and σ_e^2. Upon setting the expected values of the quadratic forms equal to the quadratic forms and solving the resulting equations, the following estimators are obtained:

$$\hat{\sigma}_e^2 = d^{-1}\underset{\sim}{W}_k'[\underset{\sim}{I} - \underset{\sim}{Z}(\underset{\sim}{Z}'\underset{\sim}{Z})^{-1}\underset{\sim}{Z}']\underset{\sim}{W}_k \quad \text{and} \quad \hat{\sigma}_b^2 = (ncd)^{-1}\underset{\sim}{W}_k'[n(h+d)\underset{\sim}{Z}(\underset{\sim}{Z}'\underset{\sim}{Z})^{-1}\underset{\sim}{Z}' - d\underset{\sim}{J} - nh\underset{\sim}{I}]\underset{\sim}{W}_k$$

where

$$h = \text{trace}\{[\underset{\sim}{Z}(\underset{\sim}{Z}'\underset{\sim}{Z})^{-1}\underset{\sim}{Z}' - \underset{\sim}{J}/n]\underset{\sim}{G}_k\underset{\sim}{G}_k'\}, \qquad c = (n - n^{-1}j'\underset{\sim}{Z}\underset{\sim}{Z}'j),$$

$$d = \text{trace}\{[\underset{\sim}{I} - \underset{\sim}{Z}(\underset{\sim}{Z}'\underset{\sim}{Z})^{-1}\underset{\sim}{Z}']\underset{\sim}{G}_k\underset{\sim}{G}_k'\}, \quad \text{and} \quad j \text{ is a vector of 1's}.$$

The distributions and the properties of these estimators are cumbersome to investigate because of rather intractible algebraic expressions, but approximate lower bounds for the mean square errors of these estimators were obtained.

$$\underset{\sim}{\text{MSE}}(\hat{\sigma}_e^2) \geq 2d^{-2}\sigma_e^2 \, \text{trace}\{[\underset{\sim}{I} - \underset{\sim}{Z}(\underset{\sim}{Z}'\underset{\sim}{Z})^{-1}\underset{\sim}{Z}']\underset{\sim}{G}_k\underset{\sim}{G}_k'\}^2$$

and

$$\underset{\sim}{\text{MSE}}(\hat{\sigma}_b^2) \geq 2(ncd)^{-2} \, \text{trace}\{[n(c+d)\underset{\sim}{Z}(\underset{\sim}{Z}'\underset{\sim}{Z})^{-1}\underset{\sim}{Z}' - d\underset{\sim}{J} - nh\underset{\sim}{I}][\sigma_b^2\underset{\sim}{Z}\underset{\sim}{Z}' + \sigma_e^2\underset{\sim}{G}_k\underset{\sim}{G}_k']\}^2 \; .$$

2. SIMULATIONS AND OBSERVATIONS

Monte Carlo evaluations were undertaken to investigate the reduction in mean square error through the use of the described procedure. The results from one of the investigations are present. A (25 x 4) $\underset{\sim}{X}$ matrix was created with the property that the numbers in Columns 1 and 2 were randomly paired. Column 3 was created by taking a linear combination of the Columns 1 and 2, altered slightly so that the $\underset{\sim}{X}'\underset{\sim}{X}$ matrix would not be singular. Column 4 was a multiple of Column 2, similarly altered. The smallest characteristic root of $\underset{\sim}{X}'\underset{\sim}{X}$ was .00008696 and of $\underset{\sim}{\mathcal{X}}'\underset{\sim}{\mathcal{X}}$ the smallest root was .00008644 . The $\underset{\sim}{X}$ matrix for the investigation is displayed in Table 1.

In the different simulations various permutation of the vectors (1,1,1,1), (1,1,5,6) and (1,1,12,13) were used as the $\underset{\sim}{\tau}$ vectors and various pairings of .25, 4 and 9 were used for the values of the variance components σ_b^2 and σ_e^2 .

Different (25 x 5) $\underset{\sim}{Z}$ matrices were selected:

(a) a balanced design matrix (5 "ones" in each of the 5 columns),

(b) an unbalanced design matrix where the number of ones in the columns were different permutations of the number 2, 3, 5, 7, 8 .

Five hundred sets of observations were generated for each configuration of $\underset{\sim}{\tau}$, σ_b^2, σ_e^2 and $\underset{\sim}{Z}$. Table 2 presents the results of these simulations as they apply to $\underset{\sim}{X}\underset{\sim}{\tau}$. Tables 3 and 4 present the results pertinent to the variance components. Table 3 contains a summary for the balanced design case and Table 4 contains a summary for the unbalanced design case. In these tables the reduction in mean square error using ridge regression is expressed as a percentage of the mean square error obtained by ordinary least squares.

The values of k presented in the tables are approximate values corresponding to the minimum $\underset{\sim}{MSE}$. Investigations showed that differences between $\underset{\sim}{MSE}$ corresponding to a $k > 1$ and $\underset{\sim}{MSE}$ for $k = 1$ were slight, hence the value $k = 1$ appears in the tables often as a replacement for a value of $k > 1$.

The tables present average estimates over sets of 500 simulations. We make the following observations relative to the estimation of $\underset{\sim}{X}\underset{\sim}{\tau}$:

1. The $\underset{\sim}{MSE}$ varies directly with the magnitude of $\underset{\sim}{\tau}'\underset{\sim}{\tau}$.
2. Reductions in $\underset{\sim}{MSE}$ were obtained in all cases.
3. Changes in the signs and permutations of the elements of $\underset{\sim}{\tau}$ did not produce sizeable variations in the magnitude of the $\underset{\sim}{MSE}$.
4. As $\underset{\sim}{\tau}'\underset{\sim}{\tau}$ increases, the value of k needed to minimize the $\underset{\sim}{MSE}$ tends toward zero.
5. If neither of the variance components are near zero, there is a range of values of k which produces near maximum reduction in $\underset{\sim}{MSE}$. Thus the choice of k is not critical.
6. Larger $\underset{\sim}{MSE}$s were obtained for the unbalanced designs and the percentage reduction in these cases were also larger.
7. For the same configuration of $\underset{\sim}{\tau}$ and the variance components, greater reductions in $\underset{\sim}{MSE}(\underset{\sim}{X}\underset{\sim}{\hat{\tau}}_k)$ were made in the balanced case than in the unbalanced configurations of $\underset{\sim}{Z}$.

Some conclusions for the estimation of the variance components follow:

1. For ordinary least squares estimation of the fixed effects, the average of the estimates $\hat{\sigma}_b^2$ was close to σ_b^2 when σ_e^2 was small but became a substantial overestimate as σ_e^2 increased.

2. On the interval (0,1) the average of the estimates $\hat{\sigma}_b^2$ was monotone decreasing in k and the average of the estimates $\hat{\sigma}_e^2$ was monotone increasing in k .

3. There always existed values of $k > 0$ which produced significant reductions in the $\underset{\sim}{MSE}$. In many cases, the value of k which minimized $\underset{\sim}{MSE}(\underset{\sim}{X}\hat{\underset{\sim}{\tau}}_k)$ did not produce the minimum mean square error for either of the variance components. The value of k which produced the minimum for σ_e^2 rarely exceeded the value of k which minimized the mean square error for the other variance component.

4. The optimal choice of k varied inversely with the magnitude of $\underset{\sim}{\tau}'\underset{\sim}{\tau}$.

5. It appears that a fairly wide range of k will produce near maximal reductions in $\underset{\sim}{MSE}$ which implies the choice of k is not critical.

6. Permutations of the values of τ resulted in large variations in the values of k which produced the minimum mean square errors of the variance components. These differences had little effect on the magnitude of the $\underset{\sim}{MSE}$ or on the magnitude of the estimates of the variance components.

7. The effect of the unbalanced design in the $\underset{\sim}{Z}$ matrix was a larger mean square error in the estimates of the variance components. The magnitudes of the estimates showed only slight susceptability to the effects of the unbalanced $\underset{\sim}{Z}$ matrix. The estimates of σ_b^2 were very slightly smaller and the estimates of σ_e^2 were slightly larger when the unbalanced design was introduced.

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TABLE 1 – $\underset{\sim}{X}$-Matrix used in the simulations

6.24	8.08	4.40	16.17
.92	.74	1.10	1.47
4.99	9.55	.42	19.10
9.02	4.08	13.95	.82
7.75	5.43	10.07	10.86
7.61	6.41	8.80	12.83
2.85	6.02	- .32	12.03
2.07	.87	3.28	1.73
9.26	2.68	15.83	5.35
2.84	5.93	- .27	11.86
9.90	8.84	10.95	17.79
9.64	5.48	13.80	10.97
3.43	5.77	1.08	11.54
3.62	8.89	-1.66	17.77
3.46	2.84	4.07	5.67
3.19	8.37	-1.99	16.74
6.62	.58	12.67	1.16
.14	5.66	-5.38	11.31
4.43	8.31	.52	16.62
.38	7.45	6.70	4.91
.70	9.69	-8.30	19.38
2.10	4.07	.14	8.13
4.52	6.10	2.94	12.20
3.49	.53	6.46	1.05
3.61	5.84	1.38	11.67

TABLE 2 — Simulation results in the estimation of $\underset{\sim}{X}\underset{\sim}{\tau}$

Parameters			Balanced Designs				Unbalanced Designs			
σ_b^2	σ_e^2	$\underset{\sim}{\tau}'\underset{\sim}{\tau}$	$\underset{\sim}{MSE}$ k=0	MIN $\underset{\sim}{MSE}$	k	% Red.	$\underset{\sim}{MSE}$ k=0	MIN $\underset{\sim}{MSE}$	k	% Red.
0.25	4.00	4	18.62	4.73	1.00	74.3	18.75	4.76	1.00	74.6
		63	18.62	9.01	.30	51.6	18.75	9.08	.30	51.6
		271	18.62	11.16	.10	40.0	18.75	11.32	.10	39.6
0.25	9.00	4	41.89	9.22	1.00	78.0	42.19	9.24	1.00	78.1
		63	41.89	16.77	.50	60.0	42.19	16.81	.50	60.2
		271	41.89	22.70	.20	45.8	42.19	22.86	.20	45.8
4.00	0.25	4	1.16	.67	.10	42.7	1.17	.75	.05	36.3
		63	1.16	.83	.05	28.4	1.17	.85	.05	27.6
		271	1.16	.86	.01	26.1	1.17	.87	.01	25.9
4.00	4.00	4	18.62	5.16	1.00	72.3	18.75	5.77	1.00	69.2
		63	18.62	7.76	1.00	58.2	18.75	8.95	.30	52.2
		271	18.62	11.03	.20	40.1	18.75	12.98	.05	30.8
4.00	9.00	4	41.89	9.61	1.00	77.1	42.19	10.23	1.00	75.8
		63	41.89	15.32	.75	63.4	42.19	14.33	.88	66.6
		271	41.89	22.01	.48	46.7	42.19	23.91	.15	43.4
9.00	0.25	4	1.16	.72	.05	38.1	1.17	.79	.05	32.3
		63	1.16	.83	.01	28.3	1.17	.85	.01	27.4
		271	1.16	.86	.01	26.1	1.17	.87	.01	25.7
9.00	4.00	4	18.62	5.81	1.00	68.8	18.75	7.15	1.00	61.9
		63	18.62	9.24	.42	50.4	18.75	9.52	.60	49.3
		271	18.62	10.87	.10	41.7	18.75	12.00	.08	36.1
9.00	9.00	4	41.89	10.23	1.00	75.6	42.19	11.60	1.00	72.5
		63	41.89	12.61	1.00	69.9	42.19	13.76	1.00	67.5
		271	41.89	21.24	.20	49.2	42.19	23.02	.20	45.5

TABLE 3 — Simulation results in the estimation of the variance components, balanced design

Parameters			Estimates of σ_b^2						Estimates of σ_e^2					
σ_b^2	σ_e^2	$\tau'\tau$	MSE k=0	MIN MSE	k	% Red.	Est. k=0	Est. R.R.	MSE k=0	MIN MSE	k	% Red.	Est. k=0	Est. R.R.
0.25	4.00	4	.73	.54	1.00	25.3	.38	.25	1.88	1.60	1.00	15.0	3.95	3.99
		63	.73	.59	.30	19.2	.38	.27	1.88	1.66	.30	11.3	3.95	4.05
		271	.73	.63	.10	14.8	.38	.30	1.88	1.71	.10	9.0	3.95	4.03
0.25	9.00	4	2.81	2.02	1.00	28.1	.53	.24	9.51	8.04	1.00	15.5	8.89	8.93
		63	2.81	2.16	.50	23.2	.53	.27	9.51	8.30	.50	12.7	8.89	9.07
		271	2.81	2.28	.20	18.6	.53	.31	9.51	8.51	.20	10.5	8.89	9.12
4.00	0.25	4	8.73	8.01	1.00	8.3	4.02	3.82	.01	.01	.10	8.2	.25	.25
		63	8.73	8.20	.50	6.1	4.02	3.85	.01	.01	.05	6.8	.25	.25
		271	8.73	8.42	.20	3.6	4.02	3.91	.01	.01	.01	5.5	.25	.25
4.00	4.00	4	10.73	—	1.00	—	4.14	—	1.94	—	1.00	—	4.01	—
		63	10.73	9.44	1.00	12.2	4.14	3.83	1.94	1.75	.20	9.8	4.01	4.09
		271	10.73	9.94	.20	7.5	4.14	3.84	1.94	1.76	.10	9.4	4.01	4.09
4.00	9.00	4	20.38	16.87	1.00	17.2	4.37	3.88	9.51	8.09	1.00	15.0	8.89	8.96
		63	20.38	17.20	.81	15.6	4.37	3.90	9.51	8.36	.66	12.1	8.89	9.11
		271	20.38	18.06	.57	13.0	4.37	3.95	9.51	8.57	.20	9.9	8.89	9.13
9.00	0.25	4	43.30	39.77	1.00	8.2	9.02	8.59	.01	.01	.10	6.8	.25	.25
		63	43.30	40.28	.75	7.0	9.02	8.62	.01	.01	.05	5.5	.25	.25
		271	43.30	41.19	.40	4.9	9.02	8.68	.01	.01	.01	5.5	.25	.25
9.00	4.00	4	53.39	47.53	1.00	11.0	9.22	8.65	1.88	1.62	4.00	13.8	3.95	4.11
		63	53.39	48.10	.80	9.9	9.22	8.67	1.88	1.69	.24	9.9	3.95	4.05
		271	53.39	49.41	.40	7.4	9.22	8.77	1.88	1.71	.15	9.0	3.95	4.08
9.00	9.00	4	67.33	58.18	1.00	13.6	9.43	8.68	9.51	8.13	1.00	14.5	8.89	8.99
		63	67.33	58.12	1.00	13.7	9.43	8.68	9.51	8.37	1.00	12.0	8.89	9.14
		271	67.33	60.65	.50	9.9	9.43	8.79	9.51	8.50	.30	10.6	8.89	9.18

TABLE 4 — Simulation results in the estimation of the variance components, unbalanced design

Parameters			Estimates of σ_b^2						Estimates of σ_e^2					
σ_b^2	σ_e^2	$\tilde{\tau}'\tilde{\tau}$	MSE $\tilde{k}=0$	MIN $\widetilde{\text{MSE}}$	k	% Red.	Est. k=0	Est. R.R.	MSE $\tilde{k}=0$	MIN $\widetilde{\text{MSE}}$	k	% Red.	Est. k=0	Est. R.R.
0.25	4.00	4	.75	.55	1.00	26.6	.37	.22	1.99	1.69	1.00	14.3	3.98	4.01
		63	.75	.59	.50	21.3	.37	.22	1.99	1.77	.30	10.8	3.98	4.07
		271	.75	.63	.20	16.1	.37	.23	1.99	1.83	.10	8.0	3.98	4.05
0.25	9.00	4	2.88	2.01	1.00	28.9	.50	.20	10.06	8.53	1.00	15.1	8.96	8.97
		63	2.88	2.15	.75	25.5	.50	.18	10.06	8.82	.75	12.3	8.96	9.25
		271	2.88	2.29	.40	20.4	.50	.16	10.06	9.08	.20	9.7	8.96	9.16
4.00	0.25	4	10.90	9.02	1.00	17.3	4.12	3.77	.01	.01	.10	7.7	.25	.26
		63	10.90	9.26	.75	15.1	4.12	3.76	.01	.01	.05	6.4	.25	.26
		271	10.90	9.64	.40	11.6	4.12	3.77	.01	.01	.01	6.4	.25	.25
4.00	4.00	4	15.66	12.66	1.00	19.2	4.26	3.78	2.00	1.72	1.00	13.5	3.98	4.06
		63	15.66	12.81	.88	18.0	4.26	3.78	2.00	1.79	.30	9.9	3.98	4.07
		271	15.66	13.40	.50	14.5	4.26	3.74	2.00	1.84	.12	7.6	3.98	4.09
4.00	9.00	4	22.90	18.13	1.00	20.8	4.41	3.78	10.06	8.58	1.00	14.7	8.96	9.02
		63	22.90	18.36	1.00	19.9	4.41	3.76	10.06	8.80	.88	13.4	8.96	9.18
		271	22.90	19.08	.88	16.7	4.41	3.68	10.06	9.16	.20	9.4	8.96	9.19
9.00	0.25	4	54.18	44.75	1.00	17.4	9.24	8.46	.01	.01	.05	6.4	.25	.25
		63	54.18	45.41	1.00	16.2	9.24	8.45	.01	.01	.01	5.1	.25	.25
		271	54.18	46.81	.50	13.6	9.24	8.49	.01	.01	.005	5.1	.25	.25
9.00	4.00	4	64.90	53.00	1.00	18.3	9.41	8.50	1.99	1.76	.50	11.3	3.98	4.06
		63	64.90	53.39	1.00	17.8	9.41	8.49	1.99	1.81	.30	9.1	3.98	4.11
		271	64.90	54.86	.75	15.5	9.41	8.40	1.99	1.84	.13	7.4	3.98	4.11
9.00	9.00	63	63.33	50.96	1.00	19.5	9.20	8.12	10.68	9.35	1.00	12.4	9.07	9.31
		271	63.33	53.54	.75	15.5	9.20	8.09	10.68	9.63	.20	9.7	9.07	9.29